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Second-Order Effects in Laminar Boundary Layers

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Second-order boundary layer disturbances are due to the displacement of the main flow by the boundary layer, surface curvature, freestream vorticity, and slip. A procedure for finding these is given for compressible flow of a perfect gas having a classically similar boundary layer. Solutions are given for the flat plate and circular cylinder and for the hypersonic axisymmetric stagnation point. For the latter flow, the dominant effect is that of vorticity, which increases both shear and heat flux. For the plate or cylinder, the same conclusion tends to hold for high speed flow. The vorticity effect is governed by the entire outer flow—not just the wall vorticity.

Nomenclature

a_1, c_1 = slip parameters, Eq. (17)
C_f = skin friction coefficient, Eq. (18a)
f, F, g, G = similarity functions, Eq. (9)
H = local stagnation enthalpy
K = constant, Eq. (9)
M = Mach number
Nu = Nusselt number, Eq. (18b)
P = pressure
Pr = Prandtl number
r = radial coordinate
R, R_0 = longitudinal radius of curvature; local and stagnation point, respectively
Re = Reynolds number, $\bar{\rho}\bar{u}\xi/\bar{\mu}$
T = temperature
u, v = velocity components along and normal to surface
x, z = distorted coordinates, Eq. (8)
γ = ratio of specific heats
Γ = profile function, Eq. (22a)
ϵ = small quantities depending on x, Eq. (9)
η = similarity variable, Eq. (8)
ζ, ξ = coordinates normal and tangent to surface
θ = $\cos^{-1}(dr_w/d\xi)$
λ_1 = function defined in Eq. (25a)
μ = viscosity
ρ = density
σ = 0 or 1 for plane or axisymmetric flow

Barred quantities are evaluated at the wall in the absence of viscosity.

Subscripts

e = conditions outside the boundary layer in its presence
w = wall
0 = first approximation

Use of a coordinate as a subscript indicates differentiation.

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Introduction

UNDER extreme conditions of fluid motion, particularly in very high speed flight at high altitude, the boundary layer near a surface can become sufficiently thick to affect materially the external flow. In turn, this influences the forces and heat transfer at the wall. At the same time and to the same order, roughly speaking, other assumptions of the classical boundary layer theory begin to break down.

The upshot is that four effects appear, all of order $Re^{-1/2}$, with respect to the classical results. These are due to the following: the boundary layer itself, nonuniformities in the external stream, curvature of the surface, and slip (in velocity and temperature) at the surface.

Two points of view can be taken. One is to consider a sharp leading edge and examine the flow "near" the edge, where the boundary layer exerts a considerable effect on the outer flow. This is the so-called "strong interaction" problem and it is not considered here. The other viewpoint, the subject of this discussion, involves weak interactions. One assumes that the boundary layer causes only small changes in the external flow. This is of interest near a blunt stagnation point and far back on a sharp-edged body.

Various aspects have been discussed by a number of authors. The self-induced effect of the boundary layer on a flat plate at high speed is described in Refs. 1–3, and at low speed in Refs. 1 and 4. The effect of freestream nonuniformities has been debated at length for the incompressible flat plate, for example, Refs. 5–10. The stream vorticity effect at a stagnation point is analyzed in Refs. 11–13. The effect of lateral curvature has been discussed by Probstein and Elliott,[14] whereas slip at the wall has been examined in Refs. 1, 15, and 16, among others.

A more complete general discussion of the stagnation point has been given by Lenard.[17] Finally, a general discussion of higher approximations has been given by Van Dyke[18,19] with particular reference to the flow near a stagnation point.

To the order currently of interest, questions of the form of the expansion (see, e.g., Goldstein[20]) or of optimal coordinates (Kaplun[21]) are not of concern. To second order (the first order understood to be the classical result), there seems to be no marked advantage of one coordinate system over any other, and the difficulties associated with the form of expansion—in particular, the introduction of logarithmic terms—arise only in higher order solutions.

Hence, in what follows, it is assumed that there are no hidden subtleties in either the coordinate system or the expansions. There is no consideration of real gas effects or other than a linear temperature-viscosity relation. Furthermore, the external flow is taken to have uniform stagnation enthalpy.

The four second-order effects of interest appear in different ways, and although the solution is found by a perturbation in both the boundary layer and the external flow—with resulting linear equations—it does not follow that one can solve for each effect separately and add them. This is not to say that linear effects cannot be added, but rather that the four effects are not clearly separable. The trouble lies in the boundary conditions applied on the perturbed viscous layer equations at the outer edge of the boundary layer. In general, the method to be pursued is that of "inner and outer expansions," as first introduced by Lagerstrom and Cole.[22]

One solves the inviscid flow and uses the "surface" values of velocity, temperature, and pressure as external boundary conditions for the boundary layer, where the Prandtl equations apply. This is the usual procedure. Next, from the boundary layer solution, one finds the normal (to the wall) velocity near the outer edge of the boundary layer. This defines a displaced stream surface, which serves to drive a perturbation (of order $Re^{-1/2}$) in the inviscid flow.

It is at this point that the various effects no longer may be separable, even though the effect is a perturbation of the first-order motion. Finally, one expands the inner flow in a series about its classical solution and applies the appropriate perturbed boundary conditions. If there is a slip at the wall, the surface conditions are altered. The remaining effects enter the boundary conditions far from the surface or, as in the present formulation, as nonhomogeneous terms in the perturbation equations of motion.

The question of the distinguishing of various effects is of sufficient importance to merit some discussion here. Vorticity and curvature cause the problem. These effects are small in the viscous layer but are generally large outside it. That is, it must be remembered that, although the perturbations engendered by the presence of the boundary layer are all of order $Re^{-1/2}$, the first-order external flow contains vorticity and curvature. Hence, the perturbation of the external flow due to the displacement effect of the boundary layer will differ with differing curvature and stream vorticity.

The reflection of this perturbation on the second-order viscous flow then also will contain these two effects. The perturbed boundary layer equations will be seen to contain nonhomogeneous terms, which depend explicitly on the curvature, the vorticity, and the induced pressure and velocity at the outside of the viscous layer. The exact way in which these terms appear depends somewhat on the form of the transformations used and on the way in which the equations are written.

One might define the curvature effect, for example, as that due to these nonhomogeneous terms in the differential equation which directly contain the curvature. However, as pointed out earlier, the curvature does influence the displacement effect, and the form in which the equations have been cast influences the way in which the curvature appears. Hence, unfortunately, the division of the effects is somewhat arbitrary (see also comment by Rott and Lenard[23] to the same effect).

General Analysis

First, the assumptions are repeated. The gas is assumed to be perfect and the external stream isoenergetic. A linear viscosity-temperature relation also has been assumed. This is a source of some dissatisfaction. However, except in a special case,[1,5] any other assumption would complicate matters unduly. A further limitation is to plane or axisymmetric motion, and the flows considered are only those for which the first-order results exhibit similarity.

If the Navier-Stokes equations are put into curvilinear coordinates (ξ, ζ), where ξ is measured along the surface and ζ normal to it, and u, v are the corresponding velocities, then these equations become

$$
\begin{aligned}
&(\rho u r^\sigma)_\xi + [\rho v r^\sigma(1 + \zeta/R)]_\zeta = 0 \\
&\rho\left[u u_\xi + v u_\zeta\left(1 + \frac{\zeta}{R}\right) + \frac{uv}{R}\right] + P_\xi = \\
&\quad \frac{1}{r^\sigma R(R+\zeta)}\left[\mu r^\sigma (R+\zeta)^2\left(u_\zeta + \frac{Rv_\xi - u}{R+\zeta}\right)\right]_\zeta + \\
&\frac{2}{r^\sigma}\left[\mu r^\sigma\left(\frac{R\,u_\xi + v}{R+\zeta}\right)\right]_\xi - \frac{2\mu\sigma}{r^2}\frac{(R+\zeta)}{R}(u\sin\theta + v\cos\theta)\sin\theta - \\
&\quad \frac{2}{3}\left\{\mu\left[\frac{Ru_\xi + v}{R+\zeta} + v_\zeta + \frac{\sigma}{r}(u\sin\theta + v\cos\vartheta)\right]\right\}_\xi \\
&\rho\left[\frac{Ruv_\xi}{R+\zeta} + v v_\zeta - \frac{u^2}{R+\zeta}\right] + P_\zeta = \\
&\quad \frac{2}{r^\sigma(R+\zeta)}[\mu r^\sigma (R+\zeta) v_\zeta]_\zeta - \frac{2\mu}{R+\zeta}\left[\frac{Ru_\xi + v}{R+\zeta}\right] + \\
&\quad \frac{R}{(R+\zeta)r^\sigma}\left[\mu r^\sigma\left(u_\zeta + \frac{Rv_\xi - u}{R+\zeta}\right)\right]_\xi - \qquad (1) \\
&\quad \frac{2\mu\sigma\cos\theta}{r^2}(u\sin\theta + v\cos\theta) - \\
&\quad \frac{2}{3}\left[\mu\left(\frac{Ru_\xi + v}{R+\zeta}\right) + \mu v_\zeta + \mu\sigma\frac{u\sin\theta + v\cos\theta}{r}\right]_\zeta \\
&\rho\left[\frac{Ruh_\xi}{R+\zeta} + v h_\zeta\right] - \left[\frac{RuP_\xi}{R+\zeta} + vP_\zeta\right] = \\
&\frac{1}{r^\sigma}\left[\left(\frac{r^\sigma \mu h_\zeta}{Pr}\right)_\zeta + \frac{R}{R+\zeta}\left(\frac{Rr^\sigma \mu h_\xi}{Pr(R+\zeta)}\right)_\xi\right] + \frac{\mu h_\zeta}{Pr(R+\zeta)} + \\
&\quad \mu\left\{2v_\zeta^2 + 2\left(\frac{Ru_\xi + v}{R+\zeta}\right)^2 + \frac{2\sigma}{r^2}(u\sin\theta + v\cos\theta)^2 + \right. \\
&\quad \left(u_\zeta + \frac{Rv_\xi - u}{R+\zeta}\right)^2 - \frac{2}{3}\left[\frac{Ru_\xi + v}{R+\zeta} + v_\zeta + \right. \\
&\quad \left.\left.\frac{\sigma}{r}(u\sin\theta + v\cos\theta)\right]^2\right\}
\end{aligned}
$$

where r is the ordinary cylindrical radius of the point (ξ, ζ); $\sigma = 0, 1$ for plane or axisymmetric flow; R is the longitudinal radius of curvature of the surface, and the remaining terms have their usual definitions.

Now assume a boundary layer. If one retains the usual terms (which lead to the familiar relations) and, in addition, the next terms in an expansion in inverse powers of $Re^{1/2}$ (Re being based on, say, stream conditions near the wall), one gets, without difficulty,

$$(\rho u r^\sigma)_\xi + [\rho v r^\sigma(1 + \zeta/R)]_\zeta = 0 \qquad (2)$$

$$\rho\left[u u_\xi + v u_\zeta\left(1 + \frac{\zeta}{R}\right) + \frac{uv}{R}\right] + P_\xi = \frac{1}{r^\sigma}\left[\mu r^\sigma u_\zeta\left(1 + \frac{\zeta}{R}\right)\right]_\zeta - \frac{\mu_\zeta u}{R} \qquad (3)$$

$$P_\zeta = \rho u^2/R \qquad (4)$$

$$\rho\left[uH_\xi + vH_\zeta\left(1+\frac{\zeta}{R}\right)\right] = \frac{1}{r^\sigma}\left[\frac{\mu r^\sigma H_\zeta}{Pr}\left(1+\frac{\zeta}{R}\right)\right]_\zeta - \frac{1}{R}(\mu u^2)_\zeta + \frac{Pr-1}{Prr^\sigma}\left[\mu r^\sigma u u_\zeta\left(1+\frac{\zeta}{R}\right)\right]_\zeta \quad (5)$$

$$P/\rho T = \text{const} \quad (6)$$

where H is the stagnation enthalpy.

Note that the stream vorticity near the surface is given by $u_\zeta + u/R$ and that, also at the surface, the following relations hold among normal derivatives of the inviscid quantities:

$$\begin{aligned} T_\zeta/T &= -(\gamma-1)M^2(u_\zeta/u) \\ P_\zeta/P &= \gamma M^2/R \\ \rho_\zeta/\rho &= (\gamma M^2/R) + (\gamma-1)M^2(u_\zeta/u) \\ S_\zeta/c_P &= -(\gamma-1)M^2[(u_\zeta/u) + (1/R)] \end{aligned} \quad (7)$$

where S is the entropy.

Next, a slightly modified Levy-Lees transformation and a modified similarity are introduced. The appropriate equations for this are

$$x = \int_0^\xi \bar{\mu}\bar{\rho}\bar{u}r_w^{2\sigma}d\xi$$
$$\eta = \frac{1}{2x^{1/2}}\int_0^\zeta \rho r^\sigma u_e d\zeta \quad (8)$$

where barred quantities refer to conditions near the surface in the absence of viscosity, whereas u_e is the actual inviscid tangential velocity. That is, $\bar{u}$ is the first-order inviscid velocity parallel to the surface, whereas the subscript e refers to the values at the edge of the viscous layer, in its presence. The stream function ψ and dimensionless total enthalpy F are

$$\psi = (2x)^{1/2}[f(\eta) + \sum_i \epsilon_i(x)g_i(\eta)]$$
$$F = \frac{H}{H_{\text{external}}} = F(\eta) + \sum_i \epsilon_i(x)G_i(\eta) \quad (9)$$

where the ϵ_i are all of order $Re^{-1/2}$.

If these transformations are applied to the equations of motion and one assumes a linear viscosity-temperature relation, the following set of equations results:

$$P_\eta = \frac{(2x)^{1/2}\bar{u}}{Rr^\sigma}f_\eta^2 \quad (10a)$$

or

$$P - P_e = \frac{(2x)^{1/2}\bar{u}}{Rr_w^\sigma}\int_\infty^\eta\left(f_\eta^2 - \frac{T}{\bar{T}}\right)d\eta \quad (10b)$$

$$f_{\eta\eta\eta} + ff_{\eta\eta} + (2x\bar{M}_x/\bar{M})(F - f_\eta^2) = 0 \quad (11)$$

$$\frac{1}{Pr}F_{\eta\eta} + fF_\eta + \frac{Pr-1}{Pr}\frac{(\gamma-1)\bar{M}^2}{1+[(\gamma-1)/2]\bar{M}^2}(f_\eta f_{\eta\eta})_\eta = 0 \quad (12)$$

$$\epsilon\left\{g_{\eta\eta\eta} + fg_{\eta\eta} + gf_{\eta\eta} + \frac{2x\bar{M}_x}{\bar{M}}(G - 2f_\eta g_\eta) + \frac{2x\epsilon_x}{\epsilon}(gf_{\eta\eta} - f_\eta g_\eta)\right\} = \frac{(2x)^{1/2}}{\bar{\rho}\bar{u}r_w^\sigma}\left(\frac{\bar{u}_\zeta}{\bar{u}} + \frac{1}{R}\right)(\eta - b - ff_\eta) + 2x\left(\frac{M_e - \bar{M}}{\bar{M}}\right)_x(f_\eta^2 - F) + \frac{(2x)^{1/2}}{\bar{\rho}\bar{u}r_w^\sigma}\frac{f_\eta T_\eta}{\bar{T}}\left(\frac{1}{R} - \frac{\bar{u}_\xi}{\bar{u}}\right) - \frac{(2x)^{1/2}}{\bar{\rho}\bar{u}r_w^\sigma}f_{\eta\eta}\left[\left(\frac{1}{R} + \frac{2\sigma\cos\theta}{r_w} + \frac{3\bar{u}_\zeta}{\bar{u}}\right)\frac{T}{\bar{T}} + \frac{\gamma\bar{M}^2}{R}f_\eta^2\right] + \frac{(2x)^{1/2}}{\bar{\rho}\bar{u}r_w^\sigma R}\left\{\left(1 + \frac{\gamma-1}{2}\bar{M}^2\right)\left(1 - \frac{2xR_x}{R} - 2x\sigma\frac{r_{wx}}{r_w}\right) + \frac{2x\bar{M}_x}{\bar{M}}[1 + (2\gamma-1)\bar{M}^2]\right\}\int_\infty^\eta(f_\eta^2 - F)d\eta - f_{\eta\eta\eta}\left\{\frac{u_e - \bar{u}}{\bar{u}} + \frac{P_e - \bar{P}}{\bar{\rho}} + \frac{(2x)^{1/2}\bar{u}}{\bar{P}Rr_w^\sigma}\left(1 + \frac{\gamma-1}{2}\bar{M}^2\right)\times\int_\infty^\eta(f_\eta^2 - F)d\eta + \frac{(2x)^{1/2}}{\bar{\rho}\bar{u}r_w^\sigma}\left(\frac{1}{R} + \frac{2\sigma\cos\theta}{r_w}\right)\int_0^\eta\frac{T}{\bar{T}}d\eta\right\} \quad (13)$$

$$\epsilon\left\{\frac{1}{Pr}G_{\eta\eta} + fG_\eta + gF_\eta + \frac{2x\epsilon_x}{\epsilon}(gF_\eta - f_\eta G) + \left(\frac{Pr-1}{Pr}\right)\frac{(\gamma-1)\bar{M}^2}{1+[(\gamma-1)/2]\bar{M}^2}(f_\eta g_\eta)_{\eta\eta}\right\} = -\frac{F_{\eta\eta}}{Pr}\left\{\frac{u_e - \bar{u}}{\bar{u}} + \frac{P_e - \bar{P}}{\bar{P}} + \frac{(2x)^{1/2}\bar{u}}{\bar{P}Rr_w^\sigma}\left(1 + \frac{\gamma-1}{2}\bar{M}^2\right)\times\int_\infty^\eta(f_\eta^2 - F)d\eta + \frac{(2x)^{1/2}}{\bar{\rho}\bar{u}r_w^\sigma}\left(\frac{1}{R} + \frac{2\sigma\cos\theta}{r_w}\right)\int_0^\eta\frac{T}{\bar{T}}d\eta\right\} - \frac{F_\eta}{Pr}\left\{\left(\frac{1}{R} + \frac{2\sigma\cos\theta}{r_w} + \frac{\bar{u}_\zeta}{\bar{u}}\right)\frac{T}{\bar{T}} + \frac{\gamma\bar{M}^2}{R}f_\eta^2\right\}\frac{(2x)^{1/2}}{\bar{\rho}\bar{u}r_w^\sigma} + \frac{(2x)^{1/2}}{\bar{\rho}\bar{u}r_w^\sigma R}\left(\frac{(\gamma-1)\bar{M}^2}{1+[(\gamma-1)/2]\bar{M}^2}\right)\left(\frac{Tf_\eta^2}{\bar{T}}\right)_\eta - \left(1 - \frac{1}{Pr}\right)\times\frac{(\gamma+1)\bar{M}^2}{1+[(\gamma-1)/2]\bar{M}^2}\left\{(f_\eta f_{\eta\eta})_\eta\left[\frac{P_e - \bar{P}}{\bar{P}} + 3\left(\frac{u_e - \bar{u}}{\bar{u}}\right) + \frac{(2x)^{1/2}\bar{u}}{\bar{P}Rr_w^\sigma}\left(1 + \frac{\gamma-1}{2}\bar{M}^2\right)\int_\infty^\eta(f_\eta^2 - F)d\eta + \left(\frac{1}{R} + \frac{2\sigma\cos\theta}{r_w}\right)\frac{(2x)^{1/2}}{\bar{\rho}\bar{u}r_w^\sigma}\int_0^\eta\frac{T}{\bar{T}}d\eta\right] + f_\eta f_{\eta\eta}\left(\frac{(2x)^{1/2}}{\bar{\rho}\bar{u}r_w^\sigma}\right)\times\left[\left(\frac{1}{R} + \frac{2\sigma\cos\theta}{r_w} + \frac{5\bar{u}_\zeta}{\bar{u}}\right)\frac{T}{\bar{T}} + \frac{\gamma\bar{M}^2f_\eta^2}{R}\right] - \frac{(2x)^{1/2}}{\bar{\rho}\bar{u}r_w^\sigma}\frac{u_\zeta}{\bar{u}}\frac{T_\eta}{\bar{T}}f_\eta^2\right\} \quad (14)$$

where, in Eq. (13), $b = \lim_{\eta\to\infty}(\eta - f)$.

The nonhomogeneous terms in Eqs. (13) and (14) appear as terms explicitly proportional to the perturbation in external velocity (or M) and pressure, to the inviscid velocity gradient normal to the surface, and to the surface curvature (both lateral and longitudinal). This does not imply any claim about a unique separation of the effects.

Because of the assumption of similarity, the flow configurations are limited to those wherein the Mach number varies as a simple power of the stretched distance (x) along the surface. Moreover, because of the dissipation term in the energy equation, one must have either $P_r = 1$, $\bar{M} \to 0$ or ∞, or $\bar{M} = $ const. This admits, among other cases, the flow past a flat plate or cylinder, cone and wedge flows, and stagnation points. In the stagnation case, the boundary layer is not similar throughout the entire subsonic region, and this must be considered. Further discussion of this appears later.

To iterate the outside flow, the normal velocity perturbation due to the boundary layer is needed. From Eqs. (8) and (9)

$$\rho v r^\sigma = -[(2x)^{1/2}f]_\xi = x_\xi[-(2x)^{-1/2}f + \rho\bar{u}r^\sigma f_\eta\zeta_x] \quad (15a)$$

The inviscid solution requires that

$$(\rho v r^\sigma)_{\text{inviscid}} = -\zeta(\rho u r^\sigma)_x x_\xi \quad (15b)$$

From these, after some effort, one finds that the change in normal velocity due to the boundary layer is, near the outer edge of the layer,

$$\Delta v = (2x)^{-1/2}\bar{\mu}\bar{u}r_w{}^\sigma \left\{-f_{\eta\eta}(0) + \left[1 + \frac{\gamma-1}{2}\bar{M}^2 + 2x\frac{\bar{M}_x}{\bar{M}}(1+(\gamma-1)\bar{M}^2)\right]\int_0^\infty (F - f_\eta{}^2)d\eta\right\} \quad (15c)$$

This drives the perturbation in the external flow and forms the boundary condition, applied at $\zeta = 0$, for that increment. The boundary conditions to be applied on the viscous problem remain. These are

$$\begin{aligned} &f(0) = f_\eta(0) = 0 && f_\eta(\infty) = 1 \\ &F(0) = F_w \text{ or } F_\eta(0) = 0 && F(\infty) = 1 \\ &g(0) = g_\eta(0) = g_\eta(\infty) = 0 && \\ &G(0) \text{ or } G_\eta(0) = 0 && G(\infty) = 0 \end{aligned} \quad (16)$$

In the case of slip flow, one requires, at $\zeta = 0$,

$$\begin{aligned} u(0) &= \mu a_1(P\rho)^{-1/2}u_\zeta \\ T(0) &= \mu C_1(P\rho)^{-1/2}T_\zeta + T_w \end{aligned} \quad (17a)$$

or,[15] for an insulated wall,

$$kT_\zeta + \mu u u_\zeta = 0$$

For the present variables, these become

$$\begin{aligned} \epsilon g_\eta(0) &= [\mu r^\sigma(\rho/2xP)^{1/2}]_w \bar{u}a_1 f_{\eta\eta}(0) \\ \epsilon G(0) &= [\mu r^\sigma(\rho/2xP)^{1/2}]_w \bar{u}c_1 F_\eta(0) \end{aligned} \quad (17b)$$

or, for an insulated wall,

$$G_\eta(0) = \frac{(1-Pr)(\gamma-1)\bar{M}^2}{1+[(\gamma-1)/2]\bar{M}^2} g_\eta(0)f_{\eta\eta}(0)$$

Finally, for later use, write skin friction and heat transfer coefficients in terms of the new variables:

$$\begin{aligned} C_f Re^{1/2} &\equiv \frac{2(\mu u_\zeta)_w}{\bar{\rho}\bar{u}^2}\left(\frac{\bar{\rho}\bar{u}\xi}{\bar{\mu}}\right)^{1/2} \\ &= (2K_2)^{1/2}[f_{\eta\eta}(0) + \sum_i \epsilon_i g_{i\eta\eta}(0)]\frac{P_w}{\bar{P}}\left(\frac{u_e}{\bar{u}}\right)^2 \end{aligned} \quad (18a)$$

$$\begin{aligned} \frac{Nu}{Re^{1/2}} &= \frac{\xi(kT_\zeta + \mu u u_\zeta)_w}{k(T_a - T_w)}\left(\frac{\bar{\rho}\bar{u}\xi}{\bar{\mu}}\right)^{-1/2} \\ &= \left(\frac{K_2}{2}\right)^{1/2}\frac{P_w u_e}{\bar{P}\bar{u}}\frac{[1+[(\gamma-1)/2]\bar{M}^2]\bar{T}}{T_a - T_w}\left\{F_\eta(0) + \sum_i \epsilon_i\left[G_{i\eta}(0) + \frac{(Pr-1)(\gamma-1)\bar{M}^2}{1+[(\gamma-1)/2]\bar{M}^2}g_{i\eta}(0)f_{\eta\eta}(0)\right]\right\} \end{aligned} \quad (18b)$$

where T_a is the adiabatic wall temperature and K_2 is a constant (for the cases examined) given by

$$x = \int_0^\xi \bar{\mu}\bar{\rho}\bar{u}r_w{}^{2\sigma}d\xi = \frac{\bar{\mu}\bar{\rho}\bar{u}r_w{}^{2\sigma}\xi}{K_2} \quad (18c)$$

Flat Plate or Circular Cylinder

In this case, M_x, $1/R$, and $\tan\theta$ are all zero. Therefore, the perturbation momentum and energy [Eqs. (13) and (14)] are decoupled, with the former to be solved first.

First, because it is the easiest, the slip effect can be given. This effect appears only in the wall boundary conditions, not in the nonhomogeneous part of the differential equations. Observe that $\epsilon \sim x^{-1/2}$, and therefore the appropriate solution of Eq. (13) is

$$\epsilon g(\eta) = [\mu r^\sigma(\rho/2xP)^{1/2}]\bar{u}a_1 f_\eta(\eta)$$

This yields $g_{\eta\eta}(0) = 0$. The energy equation, Eq. (14), can be integrated once (the integration constant is evaluated at $\eta \to \infty$), and if one evaluates the result at the wall it can be seen that [notice Eq. (17)] there is no heat transfer. Thus, in general, for a flat plate, slip does not affect either the shear or heat transfer.

This result is not new and does not require any particular assumption regarding the viscosity or conductivity—nor does it require a perfect gas.[15] However, it does depend critically on the lack of a longitudinal pressure gradient. Shen and Solomon[24] and Rott and Lenard[23] have suggested that this conclusion is incorrect. However, their argument founders on the implied assumption of constant (in ξ) gas temperature near the wall, even in the presence of slip. This is not true, because the temperature jump from the (constant) wall value varies as $\xi^{-1/2}$. The question is discussed in Ref. 25.

Next to be considered is the effect of external vorticity on the flow past a flat plate or cylinder. The induced effects associated with the boundary layer displacement and with external shear must be examined jointly, since they are not really separable. When one solves the usual boundary layer, it follows that the normal velocity near the outside of the layer [see Eq. (15a)], for $Pr = 1$, is

$$v = \bar{u}\xi^{-1/2}\{K \equiv (\bar{\mu}/\bar{\rho}\bar{u})^{1/2}[0.33[(\gamma-1)/2]\bar{M}^2 + 0.86(T_w/\bar{T})]\} \quad (19)$$

If there were no freestream vorticity, the solution would be straightforward and has been given elsewhere (Refs. 1–3, e.g.). However, when the vorticity does not vanish, the external flow change must be determined as a perturbation about a nonuniform flow. For an isoenergetic inviscid parallel flow, one finds

$$\begin{aligned} u &= u_0(\zeta) \\ v &= 0 \\ P &= P_0 \text{ (const)} \\ T &= T_0 = T_{\text{stagnation}} - [u_0{}^2(\zeta)/2C_P] \\ \rho &= P_0/RT_0 \end{aligned} \quad (20)$$

where $u_0(\zeta)$ is arbitrary. Assume a perturbation about this flow, such that $u = u_0 + \Delta u$, etc. Remember that only a flat plate or cylindrical geometry is now being considered. One can define a pseudo-stream function ϕ (and a stretched normal coordinate z) to which the perturbations in pressure and velocity are related by

$$\begin{aligned} \frac{\Delta P}{\bar{\rho}\bar{u}^2} &= -2K|\bar{M}^2 - 1|^{-1/2}\left(\phi_z - \phi\frac{\Gamma_z}{\Gamma}\right) \times \frac{(M_0{}^2 - 1)}{|M_0{}^2 - 1|}\left[\frac{\Gamma}{(\Gamma)_{z=0}}\right] \\ \frac{\Delta u}{\bar{u}} &= -\frac{\Delta P}{\bar{\rho}\bar{u}^2}\left[\frac{\bar{\rho}\bar{u}}{\rho_0 u_0}\right] - 2K\phi\frac{u_{0\zeta}}{u_0} \times \left[\frac{\bar{\rho}\bar{u}}{\rho_0 u_0}\left(\frac{M_0{}^2 - 1}{\bar{M}^2 - 1}\right)^{1/2}\frac{\Gamma}{(\Gamma)_{z=0}}\right] \\ \frac{\Delta v}{\bar{u}} &= 2K\phi_\xi\left[\frac{\bar{\rho}\bar{u}}{\rho_0 u_0}\left(\frac{M_0{}^2 - 1}{\bar{M}^2 - 1}\right)^{1/2}\frac{\Gamma}{(\Gamma)_{z=0}}\right] \end{aligned} \quad (21)$$

where the terms in brackets become unity at the surface. The conservation equations for the perturbation can be combined to yield this single equation for ϕ:

$$\phi_{\xi\xi} = \left(\phi_{zz} - \phi\frac{\Gamma_{zz}}{\Gamma}\right)\frac{(M_0{}^2 - 1)}{|M_0{}^2 - 1|} \quad (22)$$

$$\begin{aligned} \Gamma &= M_0 r^{-\sigma/2}|M_0{}^2 - 1|^{-1/4} \\ z &= \int_0^\zeta |M_0{}^2 - 1|^{1/2}d\zeta \end{aligned} \quad (23)$$

The boundary condition at $\zeta = 0$ is

$$\begin{aligned} \phi(\xi,0) &= \xi^{1/2} && \xi \geqslant 0 \\ &= 0 && \xi \leqslant 0 \end{aligned}$$

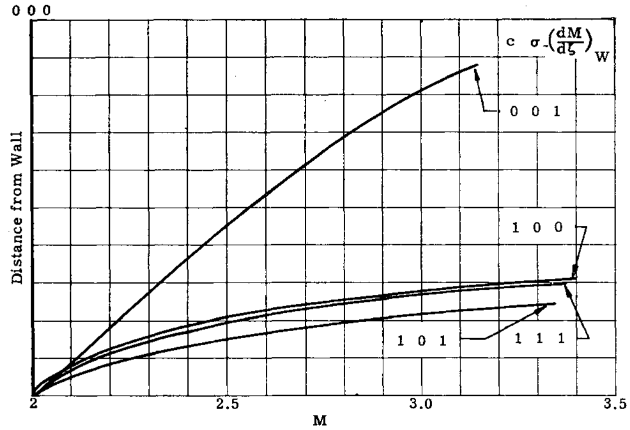

Fig. 1 Inviscid Mach number profiles from Eq. (24) using Eq. (22), M_w = 2.0

Far away the subsonic solution should decay, whereas the supersonic can admit no waves whose ultimate origin cannot be traced to the surface. Now, if

$$\Gamma_{zz} = C\Gamma \tag{24}$$

then the boundary value problem for ϕ is solved readily and leads to

$$\frac{\Delta P(\xi,0)}{\bar{\rho}\bar{u}^2} = K(\bar{M}^2 - 1)^{-1/2}\left\{\xi^{-1/2} + 2\xi^{1/2}\left(\frac{\Gamma_z}{\Gamma}\right)_{z=0} + 2C^{1/4}\int_0^{\xi C^{1/2}} \frac{\alpha^{1/2}J_1(\alpha - \xi C^{1/2})}{\alpha - \xi C^{1/2}}\,d\alpha\right\}$$

for supersonic flow, and

$$\frac{\Delta P(\xi,0)}{\bar{\rho}\bar{u}^2} = K(1 - \bar{M}^2)^{-1/2}\left\{-2\xi^{1/2}\left(\frac{\Gamma_z}{\Gamma}\right)_{z=0} + \frac{C^{1/4}}{\pi}\int_0^{\infty} \frac{(1 - \frac{4}{3}\alpha^2)K_1(\alpha - \xi C^{1/2})}{\alpha - \xi C^{1/2}}\,d\alpha\right\}$$

for subsonic flow, where J_1 is the usual Bessel function, whereas K_1 is that of the second kind of imaginary argument; K is given in Eq. (19). Also, one finds

$$C = \frac{\Gamma_{zz}}{\Gamma} = \frac{1}{4|(M_0^2 - 1)|}\left[\frac{3\sigma}{r^2} + \frac{4\sigma M_{0\zeta}}{rM_0(M_0^2 - 1)} + \frac{(10 - 3M_0^2)M_{0\zeta}^2}{(M_0^2 - 1)^2} + \frac{2(M_0^2 - 2)M_{0\zeta\zeta}}{M_0(M_0^2 - 1)}\right] \tag{24a}$$

If $M_0 \rightarrow 0$, $\sigma = 0$ (flat plate), and the velocity profile is linear then

$$\Gamma_z/\Gamma = u_\zeta/u$$

$$C \rightarrow 0$$

and

$$\Delta P/\bar{\rho}\bar{u}^2 \rightarrow -2K\xi^{1/2}u_{0\zeta}/u_0$$

$$\Delta u \rightarrow 0$$

This is the answer given by Li[6] and others. It is tempting to use Li's convenient result in a more general sense. However, this cannot be done. In general, the entire inviscid field (limited only by the customary supersonic conditions) must affect the results. For example, for the special family of profiles defined by Eq. (24), there is an effect of flow nonuniformity even when there is no shear at $\zeta = 0$.

Figure 1 depicts several such Mach number profiles. The corresponding pressures induced on the surface are given in Fig. 2. The main point, at least for these cases, is that the distribution of pressure is determined by the overall profile shape and not by the surface vorticity alone.

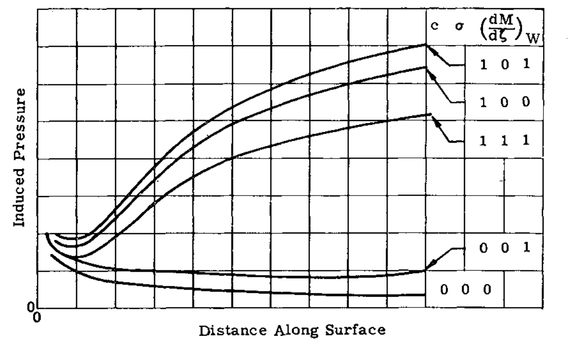

Fig. 2 Induced wall pressures for various inviscid velocity profiles

Although one can find the self-induced pressure field associated with a variety of flows, these are not used readily in the boundary layer equations for finding disturbances in the shear or heat transfer. Hence, from this point on, consider only $C = 0$. These are profiles for which there are no waves reflected back onto the surface. The applicable cases of Fig. 1 show more or less linear variation of Mach number and have (Fig. 2) falling pressures.

In this case, the perturbed boundary layers [Eqs. (13) and (14)] have nonhomogeneous terms varying as $x^{1/2}$, $x^{-1/2}$. Although a partial particular integral can be found, recourse must be had to numerical integration. After some effort, one obtains for the skin friction and heat transfer [see Eq. (18)], for $Pr = 1$,

$$\Delta C_f(Re)^{1/2} = \frac{\sigma\xi Re^{-1/2}}{r_w}\left[0.24\left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right) + 0.24 + 0.9\,\frac{T_w}{\bar{T}}\right] + \lambda_1^2[(\bar{M}^2 - 1)Re]^{-1/2} - \frac{2\lambda_1\xi}{\bar{M}^2 - 1} \times \left(\frac{\Gamma_\zeta}{\Gamma}\right)_w Re^{-1/2}\left[0.25\left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right) + 0.47(2 - \gamma\bar{M}^2) + 0.87\,\frac{T_w}{\bar{T}}\right] + \frac{2\xi\bar{u}_\zeta}{\bar{u}}Re^{-1/2}\left[-0.64\left(\frac{T_w}{\bar{T}}\right)^2 + 0.29\,\frac{T_w}{\bar{T}} + 0.83 + 0.15\left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right)^2 - 0.61\,\frac{T_w}{\bar{T}} \times \left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right) + 0.58\left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right)\right] \tag{25}$$

$$\frac{\Delta Nu}{Re^{1/2}} = \frac{\Delta C_f Re^{1/2}}{2} - \frac{\lambda_1^2}{2}[(\bar{M}^2 - 1)Re]^{-1/2} + \frac{\lambda_1\xi}{\bar{M}^2 - 1}\left(\frac{\Gamma_\zeta}{\Gamma}\right)_w Re^{-1/2}\left[0.17\left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right) + 0.47 + 0.69\,\frac{T_w}{\bar{T}}\right] + \frac{\xi\bar{u}_\zeta}{\bar{u}}Re^{1-1/2}\left[0.55\,\frac{T_w^2}{\bar{T}} - 0.89\,\frac{T_w}{\bar{T}} - 0.40 - 0.06\left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right)^2 + 0.32\,\frac{T_w}{\bar{T}}\left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right) - 0.09\left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right)\right] \tag{26}$$

where

$$\lambda_1 = 1.22(T_w/\bar{T}) + 0.47(\gamma - 1)\bar{M}^2$$

$$\left(\frac{\Gamma_\zeta}{\Gamma}\right)_w = \frac{\bar{M}^2 - 2}{2(\bar{M}^2 - 1)}\,\frac{\bar{M}_\zeta}{\bar{M}} - \frac{\sigma}{2r_w} \rightarrow \frac{\bar{u}_\zeta}{\bar{u}} \quad \text{as } M \rightarrow 0,\ \sigma = 0 \tag{26a}$$

The terms containing $(\bar{M}^2 - 1)^{1/2}$ are to be omitted for subsonic flow. Observe, using Eq. (25), that there are really no such terms in the heat transfer [Eq. (26)].

Equation (25) can be written as follows, making the perturbation more apparent. The increment in wall shear is

$$\Delta\tau = C_1\bar{\mu}\bar{u}_\zeta + (C_2\sigma\bar{\mu}\bar{u}/r_w) + (C_3\bar{\mu}\bar{u}/\xi) \qquad (27)$$

where the C's are constants, depending on $T_w/\bar{T}$ and $\bar{M}$, and the successive terms may be denoted as due to vorticity, transverse curvature, and displacement. For a particular cool wall ($T_w = \bar{T}$), some typical values follow:

M	0	0.5	2	5	10
C_1	3.11	4.5	1.0	11.3	131.5
C_2	−0.54	−0.85	+0.70	+0.34	−0.07
C_3	0.0	0.0	0.74	1.30	5.63

Observe that the effects of displacement and vorticity are large at supersonic speeds, whereas this is not the case for (transverse) curvature. The curvature affects the shear, but never by a large amount. For extreme Mach number, C_1, C_2, and C_3 vary as M^4, M^2, and M^3, respectively. Of course, the coefficient alone does not determine the relative magnitude of the effects. Similar results follow from Eq. (25) for the heat flux increment Δq. This can be written

$$\Delta q = C_4\bar{K}\bar{T}_\zeta + (C_5\sigma\bar{K}\bar{T}/R_w)$$

For $T = \bar{T}$, one has

M	0	0.5	2	5	10
C_4	−0.45	−0.60	−0.48	−5.4	−57.2
C_5	0.0	0.01	0.16	0.80	−4.6

Actually, these results are somewhat misleading. Because of the special profiles used, the shapes are not the same in two-dimensional and axisymmetric flows, even when the wall Mach number and vorticity coincide. Thus, for identical profiles, the curvature effect should also appear in the displacement term. In the limit as $M \to 0$, Eq. (27) yields $C_1 = 3.11$, which is the result given by Li.[8]

When there is no vorticity in the undisturbed stream, the present solution applies for the flat plate but not for the cylinder. For such a plate, Eqs. (25) and (26) reduce [use Eqs. (19) and (26a)] to

$$\Delta C_f Re^{1/2} = \lambda_1{}^2[(\bar{M}^2 - 1)Re]^{-1/2} \equiv \tfrac{1}{2}[Re/(\bar{M}^2 - 1)]^{1/2}(v/\bar{u})_\infty{}^2 \qquad (28)$$

$$\Delta Nu/Re^{1/2} = 0$$

In this form, the results are generally valid[1,15] and apply regardless of Prandtl number, viscosity law, or the presence of a perfect gas.

For the case of a cylinder in a uniform stream, the solution must be found by other means. Consider only the supersonic case. The normal velocity at the edge of the boundary layer is given in Eq. (19). Using this, there is a stream function ψ, for the perturbed flow, defined by

$$\psi = 2K\bar{\rho}\bar{u}r_w{}^{3/2}(\bar{M}^2 - 1)^{1/4}\int_0^{z+1-(r/r_w)} g(y)(z + 1 - y) \times \left[(z + 1 - y)^2 - \left(\frac{r}{r_w}\right)^2\right]^{-1/2} dy \qquad (29)$$

where

$$z = \xi(\bar{M}^2 - 1)^{-1/2}/r_w$$

$$g(0) = -(\tfrac{1}{2})^{1/2}$$

and $g(y)$ is found from

$$\int_0^z g_y(y)[(z + 1 - y)^2 - 1]^{1/2}dy = z^{1/2}[(1 + z/2)^{1/2} - 1]$$

This equation must be solved numerically. The result, accurate to a few percent over a reasonable range of y, is

$$g_y(y) = 0.25 - 0.10y \qquad y \leqslant 1$$
$$= 0.15/y \qquad y \geqslant 1$$

Then the resulting perturbations in streamwise velocity and pressure near the surface are

$$\left(\frac{\Delta u}{\bar{u}}\right)_w = -\left(\frac{\Delta P}{\bar{\rho}\bar{u}^2}\right)_w = \frac{2K}{r_w{}^{1/2}(\bar{M}^2 - 1)^{3/4}} \times \left\{\lambda(z) \equiv -(2z^2 + 4z)^{-1/2} + \int_0^z g_y(y)[(z + 1 - y)^2 - 1]^{-1/2}dy\right\}_{z = (\xi/r_w)(\bar{M}^2 - 1)^{-1/2}} \qquad (30)$$

The pressure distribution starts as for a flat plate, overexpands slightly, and then returns to the ambient value. If these equations are put into the differential equations for the perturbations in the boundary layer [Eqs. (13) and 14)], the resulting equations can be solved in series. This series arises because $\lambda[z(\xi)]$ is not of such a form as to permit a similar solution directly. However, with sufficient accuracy for present purposes,

$$z\lambda_z(z) = -0.02 - 0.06z^{-1/2} + (0.36/z) \qquad (z > 1) \qquad (31)$$

This permits the solution to proceed with a reasonable effort. The resulting increments in skin friction and heat transfer (for $z > 1$) are

$$\Delta C_f Re^{1/2} = \frac{\xi}{r_w} Re^{-1/2}\left[0.24 + 0.24\left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right) + 0.96\frac{T_w}{\bar{T}}\right] + 0.22\left(\frac{z}{(\bar{M}^2 - 1)Re}\right)^{1/2}\left(\frac{\gamma - 1}{2}\bar{M}^2 + 2.60\frac{T_w}{\bar{T}}\right) \times \left\{(\gamma\bar{M}^2 - 1)\left(\int_0^z \lambda dz\right)/z + 2\lambda(2 - \gamma\bar{M}^2) - \left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right)\left[0.04(1 + 3.44F_w) + \frac{0.24}{z^{1/2}}(1 + 2.60F_w)\right]\right\} \qquad (32)$$

$$\frac{\Delta Nu}{Re^{1/2}} = \frac{1}{2}\Delta C_f Re^{1/2} + \frac{1}{2}\left[\frac{z}{Re(\bar{M}^2 - 1)}\right]^{1/2}\left(\frac{\gamma - 1}{2}\bar{M}^2 + 2.60\frac{T_w}{\bar{T}}\right) \times \left[-0.44\lambda + \left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right)\left(0.02 + 0.06F_w + \frac{0.05 + 0.14F_w}{z^{1/2}} + \frac{0.15 - 0.06F_w}{z}\right)\right] \qquad (33)$$

where $\lambda(z)$ is defined in Eq. (30), $F_w = \{T_w/\bar{T}[1 + [(\gamma - 1)/2]\bar{M}^2]\}$, and $z = \xi(\bar{M}^2 - 1)^{-1/2}/r_w$.

In the limit of large Mach number and small ξ, these results should approach those for the flat plate. However, Eqs. (32) and (33) depend on Eq. (30), which is invalid for small z. For large z, that is, far from the leading edge but with M moderate, the effect of curvature dominates that of displacement.

Some of the terms in Eq. (32), associated with curvature [the first line of right side of Eq. (32)] can be compared with those found by Probstein and Elliott [Eq. (46) of Ref. 14]. Agreement is within a few percent. Observe also that these terms are identical with the first term on the right side of Eq. (25), but not to the entire curvature effect as written in Eq. (27), where the contribution from Γ_ζ/Γ is included. One might also say that the effect of curvature is the difference between the solutions for a flat plate [Eq. (28)] and a cylinder [Eqs. (32) and (33)]. All this means simply that the terms "curvature effect" or "vorticity effect" should not be understood to be defined very precisely here.

Stagnation Point in Hypersonic Flow

In the stagnation region, or at least in that part with which this paper is concerned, the flow is subsonic. In the limit, as the stagnation point is approached, one has

$$\bar{\rho},\ \bar{\mu} \rightarrow \text{const}$$

$$r_w \approx \xi \qquad \theta \approx (\pi/2) - (\xi/R_0)$$

$$\bar{u},\ \bar{M},\ \propto \xi$$

Hence

$$x \approx \bar{\mu}\bar{\rho}\bar{u}r^{2\sigma}\xi/2(\sigma + 1)$$

$$T/\bar{T} \approx F$$

and

$$2xr_x/r = 2x\bar{u}_x/\bar{u} = 2x\bar{M}_x/\bar{M} = 1/(\sigma + 1)$$

$$2xR_{0x}/R_0 = 0$$

Further, $(Pe - \bar{P})/\bar{P}$ must be of order $\bar{M}^2$ and hence negligible where it appears explicitly in the perturbed boundary layer calculations. It does not mean that the induced pressure is negligible. For example, in Eq. (13), the term

$$2x[(M_e - \bar{M})/\bar{M}]_x(f_\eta^2 - F)$$

really involves the induced pressure and is not negligible. If these limits are used, the equations for the second-order effects [Eqs. (13) and (14)] can be simplified somewhat. The still unknown quantity is the induced tangential velocity $(u_e - \bar{u})/\bar{u}$. To find this, the first-order boundary layer in the entire subsonic region is required so that the perturbed external flow can be determined. This need not be done with great accuracy, because, for practical cases, the wall will be cold and the boundary layer thin. Consequently, the induced effects will be small.

Consider a spherical cap only. It will be assumed that the perturbed flow corresponds closely to the inviscid flow past another sphere of slightly different radius whose center of curvature is displaced. The inviscid flow about a sphere is described by a stream function of the form

$$\psi = R\psi(\xi/R,\ \zeta/R,\ k)$$

where k describes the freestream conditions. For ζ/R small, this is

$$\psi = R\left[\frac{\zeta}{R} f_1\left(\frac{\xi}{R}, k\right) + \frac{\zeta^2}{2R^2} f_2\left(\frac{\xi}{R}, k\right) + \ldots\right] \quad (34)$$

Now the actual cap is of radius R_0. Consider a second cap of radius $R_1 = R_0(1 + \alpha + \beta)$, the center of which is displaced downstream by an amount $R_0\alpha$. Then the coordinates in the second case (ξ_1, ζ_1) are related to the original ones (ξ, ζ) for small α, β by

$$\begin{aligned} \zeta_1/R_1 &= (\zeta/R_0) - \beta - \alpha[1 - \cos(\xi/R_0)] \\ \xi_1/R_1 &= (\xi/R_0) - \alpha \sin(\xi/R_0) \qquad (35) \\ r_1/r &= 1 + \beta + \alpha[1 - \cos(\xi/R_0)] \end{aligned}$$

Then the velocities in the (ξ, ζ) system for flow against the R_1 body are

$$\bar{\rho}vr = -R_0(\bar{\rho}\bar{u}r)_\xi \left\{\frac{\zeta}{R_0} - \beta - \alpha\left[1 - \cos\frac{\xi}{R_0} + \frac{(\bar{\rho}\bar{u}r)}{R_0(\bar{\rho}\bar{u}r)_\xi} \sin\frac{\xi}{R_0}\right]\right\} \quad (36)$$

$$\bar{\rho}ur = f_1 + \frac{\zeta}{R_0} f_2 - \alpha R_0 \sin\frac{\xi}{R_0}\frac{df_1}{d\xi} - \left[\alpha\left(1 + \cos\frac{\xi}{R_0}\right) + \beta\right](f_1 + f_2) \quad (37)$$

On the other hand, the normal velocity v in the same region can be found from a viscous analysis. This requires the distribution of v throughout the subsonic region. In the present case, a two-term Blasius series (see Schlichting[26]) was used. A stream function ψ [compare Eq. (9)] is written as

$$\psi = (2x)^{1/2}[f_1(\eta) + x^{1/2}f_2(\eta) + \ldots] \quad (38)$$

and similarly for F. The one has for v, from Eq. (15),

$$\bar{\rho}vr \rightarrow R_0(\bar{\rho}\bar{u}r)_\xi \left\{\frac{\zeta}{R_0} - \frac{(2x)^{-1/2}}{R(\bar{\rho}\bar{u}r)_\xi}\left[\eta - f_1 - 2x^{1/2}f_2 - \int_0^\infty (1 - F_1 - x^{1/2}F_2)d\eta + \frac{\gamma - 1}{2}\bar{M}^2\left(1 + \frac{4x\bar{M}_x}{\bar{M}}\right) \times \int_0^\infty (F_1 + x^{1/2}F_2 - f_{1\eta}^2 - 2x^{1/2}f_{1\eta}f_{2\eta})d\eta\right]\right\} \quad (39)$$

Everything in Eq. (39) is known, so that comparison with Eq. (36) yields values of the unknown parameters α and β. These are found in a roughly averaged sense since, of course, the two expressions will not coincide over the whole subsonic region. However, this agreement is reasonable in the cases actually computed. So determined, the values of α, β then can be put into Eq. (37), which becomes, near $\xi = 0$,

$$\Delta u/\bar{u} = [(2x)^{1/2}/\bar{\rho}\bar{u}r^\sigma]\frac{\bar{u}_\zeta}{\bar{u}}(\eta - b) - [(\sigma + 1)\alpha + \beta] \quad (40)$$

where $\eta - b$ is the limit of the usual stagnation point similarity function f_1 [Eq. (38)] for large η. The form shown contains σ, although the argument has been presented only for axisymmetry. For plane flow, one has $u_\zeta/\bar{u} = 0$.

A similar analysis for the pressure leads to

$$\Delta P/\bar{P} = \gamma\bar{M}^2(\alpha - \beta)$$

Observe that at the stagnation point, where Eq. (40) is valid, $\Delta u/\bar{u}$ and $\bar{u}_\zeta/\bar{u}$ are independent of ξ. Numbers for $\bar{u}_\zeta/\bar{u}$, $\bar{u}$, etc., can be found from a variety of sources (for example, Hayes and Probstein[27]).

The equations for the perturbed boundary layer must be solved numerically. For an insulated wall with $Pr = 1$, the increment in skin friction is

$$\Delta C_f Re^{1/2} = \frac{2\xi}{Re^{1/2}}\Bigg[\underbrace{-\frac{0.39}{R_0}}_{\text{curvature}} + \underbrace{0.65\left(\frac{\bar{u}_\zeta}{\bar{u}} + \frac{1}{R_0}\right)}_{\text{vorticity}} - \underbrace{\gamma^{1/2}a_1(\bar{M}_\xi)_{\xi=0}}_{\text{slip}}\Bigg] \quad (41)$$

The vorticity effect is the same as that given by Rott and Lenard.[11] For a cold wall ($T_w = 0$), one obtains

$$\Delta C_f Re^{1/2} = \frac{2\xi}{Re^{1/2}}\left[-\frac{0.65}{R_0} + 0.78\left(\frac{\bar{u}_\zeta}{\bar{u}} + \frac{1}{R_0}\right) - 0.46(a_1 - c_1)(\gamma F_w)^{1/2}(\bar{M}_\xi)_{\xi=0}\right] \quad (42)$$

$$\frac{\Delta Nu}{Re^{1/2}} = \frac{2\xi}{Re^{1/2}}\left[-\frac{0.02}{R_0} + 0.13\left(\frac{\bar{u}_\zeta}{\bar{u}} + \frac{1}{R_0}\right) + 0.18(a_1 - c_1)(\gamma F_w)^{1/2}(\bar{M}_\xi)_{\xi=0}\right] \quad (43)$$

where the coefficients in the slip term are evaluated for $F_w = 0$ and are retained only to indicate the magnitude of the slip effect. The coefficients a_1 and c_1 are of order 1 and depend on the model assumed for the gas-surface interaction.

To compare the various effects, one can use the approximate constant density inviscid result given by Hayes and Probstein.[27] For the cold wall [from Eqs. (42) and (43)], one gets (using a_1, c_1 from Ref. 1, high Mach number, and $\gamma = 1.4$, $\epsilon = \frac{1}{6}$)

$$\Delta C_f Re^{1/2} = (2\xi''/R_0 Re^{1/2}) \quad [\underset{\text{curvature}}{-0.65} \quad \underset{\text{vorticity}}{+4.68} \quad \underset{\text{slip}}{-0.16F_w^{1/2}}] \quad (44)$$

$$\Delta Nu/Re^{1/2} = (2\xi/R_0 Re^{1/2})[-0.02 \quad +0.78 \quad +0.06F_w^{1/2}]$$

The numbers for the effect of vorticity agree closely with Cheng[28] and with unpublished results of Probstein and Kemp (supplied by Probstein).

Remember that the vorticity contains effects of both velocity gradient and surface curvature [Eq. (41)]. Agreement of Eq. (44) with Van Dyke[19] is not very good. Van Dyke's calculation is for $Pr = 0.7$, $F_w = 0.2$. It neglects the displacement effect. In Eqs. (41–44), the displacement effect is lumped into the curvature term and, for the cold wall, is extremely small.

The sign of the slip terms depends on the values used for a_1 and c_1. Quite reasonable values actually would indicate a changed sign—the effect of temperature slip c_1 dominating that of velocity slip a_1. However, the main point about that term is that it is very small relative to the vorticity term, and an accurate representation, therefore, is not critical.

The result is that the primary effect of second-order quantities is that of vorticity acting directly (not through the displacement) and that this increases both the shear and heat transfer. This is gratifying from a computational point of view, because it means that it is not really necessary to recompute the inviscid stagnation point flow.

Concluding Comments

A method has been prepared for finding second-order boundary layer effects on bodies wherein similarity can be applied. Numerical results for a flat plate and cylinder and for an axially symmetric stagnation point are presented. To the order of the solutions, the expansions involved behave well; the boundary layer solutions decay exponentially, and no logarithmic terms are required.

In the cases solved, the dominant effect, particularly for hypersonic speeds, is that of stream vorticity. Also, it is shown that the interaction with a nonuniform mainstream is a function of the whole profile and is not always determined primarily by the vorticity near the wall. This is indeed unfortunate, because—to calculate the second-order effects—one may be required to have a detailed knowledge of the external flow field and its perturbation. This conclusion is not necessarily valid for a cold wall, because in that case the displacement effect is so small that the external flow perturbation may be negligible.

It would be desirable to apply the method to a less restrictive situation involving a pressure gradient but admitting a similar boundary layer throughout the body length. Such a case is the hypersonic slender body. The perturbations of the inviscid stream due to a boundary layer have been found by Mirels.[29] However, the solution near the surface has serious deficiencies. The blast theory (Ref. 30, for example) predicts a vanishing density at the surface. One may be able to correct this in the manner of Sychev[31] or Yakura,[32] whose work allows for the effect of the entropy layer. It is hoped to extend the present analysis to this case.

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